EPFL - Autumn 2024	Dr. Leonid Monin
Analysis III SV MT	**Correction**
Serie de Révision	

Exercise 1.
We apply the change of variable by using the parameterization of the curve:

$$\int_\Gamma F \cdot dl = \int_{[0,1]} F(\gamma(t)) \cdot \gamma'(t)\, dt,$$

where:

$$\begin{aligned} \gamma'(t) &= \left(1,\, 6t^2,\, 9t^8\right), \\ F(\gamma(t)) &= \left(6t^3 + \cos(t + t^{18}), 3t + 2t^3 + t^9 - 1, 2t^3 + 2t^9 \cos(t + t^{18})\right). \end{aligned}$$

Thus:

$$F(\gamma(t)) \cdot \gamma'(t) = (1 + 18t^{17}) \cos(t + t^{18}) - 6t^2 + 24t^3 + 12t^5 + 24t^{11}.$$

Finally, the integral is equal to:

$$\int_\Gamma F \cdot dl = \sin 2 + 8.$$

Exercise 2.
Let us check the possible answer one-by-one in order to identify the correct one.

- Divergence. False.

$$\begin{aligned} \frac{\partial F_x}{\partial x} &= \frac{d}{dx}\Big(g\big(\sqrt{x^2+y^2+z^2}\big)\, x\Big) \\ &= \frac{x^2}{|(x,y,z)|} g'(|(x,y,z)|) + g(|(x,y,z)|), \\ \frac{\partial F_y}{\partial y} &= \frac{y^2}{|(x,y,z)|} g'(|(x,y,z)|) + g(|(x,y,z)|), \\ \frac{\partial F_z}{\partial z} &= \frac{z^2}{|(x,y,z)|} g'(|(x,y,z)|) + g(|(x,y,z)|). \end{aligned}$$

  Thus:

$$\operatorname{div} F(x,y,z) = |(x,y,z)| g'(|(x,y,z)|) + 3g(|(x,y,z)|),$$

  which is not equal to 0 in general (simple counter example: $g := 1$).

- Integral. False.

$$\int_\Gamma F \cdot dl = \int_{[0,1]} F(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Additionally, we have:

$$\begin{aligned} F(\gamma(t)) &= \big(g(|\gamma(t)|)\cos 2\pi t,\ g(|\gamma(t)|)\sin 2\pi t,\ 0\big) \in \mathbb{R}^3 \\ &= \big(g(1)\cos 2\pi t,\ g(1)\sin 2\pi t,\ 0\big). \end{aligned}$$

We can consider the case where $g(1) = 0$. In this case, we would have:

$$\int_\Gamma F \cdot dl = 0 \neq 1.$$

- Potential. True.
We can view the scalar field $G : \mathbb{R}^3 \to \mathbb{R}$, as $G(x, y, z) = h(|x, y, z|)$, where function $h : \mathbb{R} \to \mathbb{R}$ is defined by:

$$h(a) = \int_{-3}^{a} t\, g(t)\, dt.$$

Interestingly, we have: $h'(a) = a\, g(a)$.

Let us compute the gradient of G:

$$\begin{aligned} \frac{\partial G}{\partial x}(x, y, z) &= \frac{x}{|x, y, z|} h'(|x, y, z|) = x\, g(|x, y, z|), \\ \frac{\partial G}{\partial y}(x, y, z) &= y\, g(|x, y, z|), \\ \frac{\partial G}{\partial z}(x, y, z) &= z\, g(|x, y, z|), \end{aligned}$$

and thus:

$$\operatorname{grad} G(x, y, z) = g(|x, y, z|)(x, y, z) = F(x, y, z).$$

This answer is correct.

Exercise 3.
We apply the Divergence theorem:

$$\int_{\partial\Omega} F \cdot \nu\, ds = \int_\Omega \operatorname{div} F\, d\Omega.$$

Here, the divergence applied to the defined vector field is:

$$\operatorname{div} F(x, y, z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 + 0 + 0 = 0.$$

Thus:

$$\int_{\partial\Omega} F \cdot \nu\, ds = 0.$$

Exercise 4.
Function f is continous over $\mathbb{R}$ and is even. As a result, we have $Ff = f$, and the only wrong statement is "Ff consists uniquement en fonctions de sinus".

Exercise 5.
Function f looks very similar to a Fourier series. Let us manipulate its expression:

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{+\infty} \frac{1}{3^{|n|}} \cos(2\pi n x) \\
&= \sum_{n=-\infty}^{-1} \frac{1}{3^{|n|}} \cos(2\pi n x) \quad + \quad \cos(2\pi 0 x) \quad + \quad \sum_{n=1}^{+\infty} \frac{1}{3^{|n|}} \cos(2\pi n x) \\
&= \sum_{n=1}^{\infty} \frac{1}{3^{|-n|}} \cos(2\pi(-n)x) \quad + \quad 1 \quad + \quad \sum_{n=1}^{+\infty} \frac{1}{3^{n}} \cos(2\pi n x) \\
&= 1 + \sum_{n=1}^{+\infty} \frac{2}{3^{n}} \cos(2\pi n x).
\end{aligned}$$

It is now possible to identify the real Fourier coefficients directly:

$$a_0 = 1, \quad a_n = \frac{2}{3^n}, \quad b_n = 0.$$

By applying Parseval's identity, we get:

$$\begin{aligned}
\int_{[0,1]} f^2(x)\,dx &= a_0^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(a_n^2 + b_n^2) \\
&= 1 + \frac{1}{2}\sum_{n=1}^{+\infty}\left(\frac{2}{3^n}\right)^2 \\
&= 1 + 2\sum_{n=1}^{+\infty}\left(\frac{1}{9}\right)^n.
\end{aligned}$$

We can now identify the limit of a geometric series which converges:

$$\sum_{n=0}^{+\infty}\left(\frac{1}{9}\right)^n = \frac{1}{1-1/9} = \frac{9}{8},$$

and therefore:

$$\int_{[0,1]} f^2(x)\,dx = 1 + 2(9/8 - 1) = \frac{5}{4}.$$

Exercise 6.
A simple change of variable ($x = 3y$) leads to:

$$\int_{-\infty}^{+\infty} \frac{1}{1+9y^2}\, dy = \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\frac{1}{3}\, dx = \frac{1}{3}\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = \frac{1}{3}\int_{-\infty}^{+\infty} f(x)\, dx.$$

Let us recall the expression of the Fourier transforms:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-i\xi t}\, dt,$$

which leads to:

$$\int_{-\infty}^{+\infty} f(x)\, dx = \sqrt{2\pi}\hat{f}(\xi = 0) = \sqrt{2\pi}\sqrt{\frac{\pi}{2}}e^{-|0|} = \pi.$$

Therefore:

$$\int_{-\infty}^{+\infty} \frac{1}{1+9y^2}\, dy = \frac{\pi}{3}.$$

Exercise 7.
We have:

$$\mathcal{F}(g' * f)(\alpha) = \sqrt{2\pi}\mathcal{F}(g')(\alpha)\mathcal{F}(f)(\alpha) = \sqrt{2\pi}(i\alpha)\mathcal{F}(g)(\alpha)\mathcal{F}(f)(\alpha).$$

As a result: $\mathcal{F}(g' * f)(0) = 0$.

Exercise 8.
Applying Fourier transform to the equation gives:

$$-(i\alpha)^2\hat{v}(\alpha) \ + \ \lambda\hat{v}(\alpha) \ = \ \sqrt{2\pi}\hat{w}(\alpha)\hat{v}(\alpha).$$

Let us consider the different possible answer:

- $\hat{w}(0) = \frac{\lambda}{\sqrt{2\pi}}$. False.

  At $\alpha = 0$ we get:

  $$\lambda\hat{v}(0) = \sqrt{2\pi}\hat{w}(0)\hat{v}(0) \quad \rightarrow \quad \hat{v}(0)\big(\sqrt{2\pi}\hat{w}(0) - \lambda\big) = 0.$$

  Thus, $\hat{v}(0) = 0$ and/or $\hat{w}(0) = \lambda/\sqrt{2\pi}$. Because the second equation is verified here, the first one is not mandatory.

- $\hat{w}(0) = \frac{1}{\sqrt{2\pi}}$. False.

  We reuse the previous result. In the particular case $\lambda = 1$, there is no need for $\hat{v}(0) = 0$.

- $\hat{w}(\sqrt{2\pi}) = \sqrt{2\pi}$. False.

  At $\alpha = \sqrt{2\pi}$ we get:

  $$2\pi\hat{v}(\sqrt{2\pi}) + \lambda\hat{v}(\sqrt{2\pi}) = \sqrt{2\pi}\hat{w}(\sqrt{2\pi})\hat{v}(\sqrt{2\pi}) \quad \rightarrow \quad \lambda\hat{v}(\sqrt{2\pi}) = 0.$$

  In the particular case $\lambda = 0$, there is no need for $\hat{v}(\sqrt{2\pi}) = 0$.

- $\hat{w}(2\pi) = \sqrt{2\pi}$. True.

  At $\alpha = 2\pi$ we get:

  $$4\pi^2\hat{v}(2\pi) + \lambda\hat{v}(2\pi) = \sqrt{2\pi}\hat{w}(2\pi)\hat{v}(2\pi) \quad \rightarrow \quad \hat{v}(2\pi)(4\pi^2 - 2\pi + \lambda) = 0.$$

  Since $\lambda > 0$, we cannot have $\lambda = 2\pi - 4\pi^2$. Therefore we need $\hat{v}(2\pi) = 0$.